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The spectral function of random resistor networks

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Abstract. The effective complex conductivity σ_{eff} of a two-component material can be conveniently expressed as an integral transformation of a spectral function. The spectral function depends only on the geometry of the material, and can be used to calculate σ_{eff} for any particular choice of component conductivities. This is a very useful feature if the component conductivities can be varied (by changing the temperature or frequency, for example) at a fixed geometry. We present a derivation of the spectral function that identifies it as a *density of states*. We have made direct numerical calculations of the spectral function of two-dimensional random resistor networks. Two-dimensional discrete resistor networks are ideal for this study, as the Y–Δ transformation can be used as an algorithm to obtain the most detailed results to date. We identify the structure in the spectral function with clusters in the network. We give analytic expressions for the first five moments of the spectral function, which are identified as the expansion coefficients of the effective conductivity in weak-scattering theory, and compare these expressions with the moments calculated from the simulations.

1. Introduction

We consider the problem of calculating the effective conductivity of a two-component composite material. If the components have conductivity σ_1 and σ_2, which can in general be complex, then the effective conductivity σ_{eff} is a function of the geometry G of the composite, and of σ_1 and σ_2. Calculating this effective conductivity is a classic problem that has been studied extensively for many years [1] but which still remains the subject of fruitful research.

As a consequence of the linearity of Maxwell's equations, the effective conductivity σ_{eff} will be a homogeneous function of degree one in σ_1 and σ_2 and can be written as a function of the ratio $u = \sigma_2/\sigma_1$. In a series of papers on the conductivity of continuum systems, Bergman [2, 3] argued that σ_{eff} is analytic for all values of u except for isolated poles when u is real and negative. He also argued that the location of these poles is determined by the geometry of the composite, and introduced the idea of a characteristic geometric function. These ideas were developed independently by Milton [4, 5, 6] who also showed that the singularities need not be simple poles. The formalism was subsequently placed on a more rigorous footing by Golden and Papanicolaou [7]. By the Herglotz theorem in analytic function theory [8], a complex function is completely determined by the location

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and residues of its poles so it is natural to write the effective conductivity using a spectral representation

$$m(s, G) = \frac{\sigma_{eff}(\sigma_1, \sigma_2, G)}{\sigma_1} = 1 - \int_0^1 \frac{h(x, G)\ \mathrm{d}x}{s - x} \tag{1}$$

where we introduce the new variable

$$s = 1/(1 - u)$$

for the convenience of restricting the singularities of σ_{eff} to a finite interval, the positive real axis between 0 and 1. In this paper we will use the variable x for this line segment in the complex s-plane. G represents all the relevant geometrical information from the composite, $m(s, G)$ is the normalized effective conductivity and $h(x, G)$ is the spectral function.

This spectral representation is conceptually very appealing because it summarizes all the information about the geometry of the composite in a single real function defined on a finite interval. Despite this, applications to date have been mostly limited to calculating bounds [5, 6, 9] on the effective conductivity. There have not been many attempts to calculate the spectral function directly and most of the calculations that do exist in the literature are for continuum systems and limited to a few isolated defects [10, 11] or approximate calculations for simple geometries like regular arrays of spheres [3, 12, 13]. The one calculation for discrete lattices [14] is for very small systems and does not adequately display the structure of the spectral function. Considering the central role that the random resistor network has played in the theory of disordered materials, it seems appropriate to present a detailed calculation of the spectral function for this system.

Theoretical work on spectral functions has had little impact on experimental analysis to date. This is because accurate model calculations have not been possible, and also because the spectral function would be very hard to extract from experiments. There have however been some limited attempts to postulate an analytic form for the spectral function and compare the predicted effective dielectric constant with the measured effective dielectric constant of brine-saturated rocks [15, 16]. The concept of reconstructing the spectral function from experimental data is very appealing because the spectral function accounts for changes in σ_{eff} in a two-component mixture as σ_2/σ_1 varies due to changes in the external temperature or the driving frequency or both. Note that if either σ_1 or σ_2 is complex, then σ_{eff} is also complex. It is hoped that further theoretical developments will eventually lead to applications involving spectral functions in the analysis of the conductivity of real two-component composites.

In this paper we present a discussion of the spectral function of two-component random resistor networks in two dimensions (2d), for the cases of bond substitution and site substitution on a square lattice. We divide the discussion into three parts. In the next section we discuss the physical origin of the spectral function and give a defining equation. In section 3, we present a theoretical overview of the spectral function in the context of the random resistor network and derive a number of properties of the spectral function based only on symmetry and duality. We calculate the spectral function in the low-concentration limit, and use this result to develop an effective-medium theory. We discuss the moments of the spectral function, which are the expansion coefficients in weak-scattering theory, and present exact calculations of the first few moments. In section 4, we present numerical calculations of the spectral function for the complete range of concentrations and compare the numerical results with theoretical predictions.

2. The spectral function of random lattices

We consider the problem of determining the effective conductivity σ_{eff} and the spectral function for conductances randomly distributed on a 2d square net with two different distribution geometries. In the first geometry, which we refer to as *random bond substitution*, each bond is randomly assigned a conductance σ_1 with probability $1-p$ and a conductance σ_2 with probability p. In the second geometry, referred to as *random site substitution*, each site is assigned label *one* with probability $1-p$ and label *two* with probability p. Associated with each site of type *one* are four conductances σ_1, along each of the four bonds, and similarly conductances of σ_2 for sites of type *two*. The conductance of any given bond in the lattice will thus consist of two conductances connected in series, and will depend on the type of site at each end of the bond. Specifically, the conductance of bonds between neighbouring *one* sites is $\sigma_1/2$, the conductance between neighbouring *two* sites is $\sigma_2/2$, and the conductance of a bond connecting a *one* site to a *two* site is $\sigma_b = \sigma_1\sigma_2/(\sigma_1+\sigma_2)$. Note that this results in *three* types of bond in the lattice, but the system is still properly regarded as having two components.

For a given lattice substitution rule the geometry G is completely determined by the variable p, the fraction of the second phase. The poles of the normalized effective conductivity $m(s,p)$ correspond to the resonances of the network. Physically this is most easily understood by considering a network of pure capacitors $\sigma_1 = \mathrm{i}\omega C$ and inductors $\sigma_2 = 1/(\mathrm{i}\omega L)$. Then $u = \sigma_2/\sigma_1$ is real and negative, and σ_{eff} will diverge at the resonance frequencies of the system. The singularities are confined to u real and negative because if either σ_1 or σ_2 has any resistive component (a real part), u will have an imaginary component, the system will be damped and σ_{eff} will not diverge. The variable u positive and real corresponds to a network of positive resistors, which obviously has no singularities. The position of the poles depends only on the ratio σ_2/σ_1 and not on the separate values of σ_1 and σ_2, so the location of the resonances could also be understood on the basis of the less physical model of a network where σ_1 and σ_2 are respectively positive and negative resistors.

For a *finite* random network the number of poles will be finite, and the effective conductivity is therefore

$$m(s,p) = \frac{\sigma_{eff}(s,p)}{\sigma_1} = 1 - \sum_n \frac{a_n}{(s-s_n)} \tag{2}$$

where the a_n are the residues at the poles s_n. In the thermodynamic limit where N, the number of nodes, tends to infinity, the poles are *smeared out* according to some positive measure [7, 17] and we have the integral representation

$$m(s,p) = \frac{\sigma_{eff}(s,p)}{\sigma_1} = 1 - \int_0^1 \frac{h(x,p)\,\mathrm{d}x}{(s-x)} \tag{3}$$

where $h(x,p)$ is the spectral function of interest. The spectral function is a positive real function that is zero everywhere outside the real interval $x \in [0,1]$ but s can have any complex value. From the definition of equation (3) it is straightforward to calculate the spectral function by noting that

$$h(x) = \frac{1}{\pi} \lim_{\epsilon\to 0} \mathrm{Im}\left[m(x+\mathrm{i}\epsilon, p)\right] \tag{4}$$

where $x \in [0,1]$ and ϵ are real. Thus, if we can calculate $\sigma_{eff}(s,p)$ for complex s, we can obtain the spectral function. In general this is a very difficult problem, because most algorithms for finding effective properties involve minimizing an energy functional. When

the energy functional is complex, the search for a minimum becomes the search for a stationary point which is in general much harder. However, for resistor networks in two dimensions there is a very efficient algorithm (using the Y–Δ transformation [18] which is algebraically identical for real and complex impedances) to calculate the effective resistance of the network.

3. Theory

3.1. Geometry

For uncorrelated random lattice problems, where the geometry is completely determined by the variable p, we have the exchange relation

$$\sigma_{eff}(\sigma_1, \sigma_2, p) = \sigma_{eff}(\sigma_2, \sigma_1, 1-p). \tag{5}$$

If the spectral function contains all the information about the geometry necessary to calculate the effective conductivity, there must obviously be some simple mapping between $h(x, p)$ and $h(1-x, 1-p)$ where we have used the identity

$$s = \frac{1}{1-\sigma_2/\sigma_1} = 1 - \frac{1}{1-\sigma_1/\sigma_2} \tag{6}$$

which is true for all s and in particular when s is real. The two spectral functions are not simply equal because of the asymmetric way in which σ_1 appears in the definition of $h(x, p)$ in equation (3). From equation (5) we have

$$\begin{aligned} m(s, p) &= 1 - \int_0^1 \frac{h(x, p)\ \mathrm{d}x}{s-x} = \frac{\sigma_2}{\sigma_1} m(1-s, 1-p) \\ &= \left(1 - \frac{1}{s}\right)\left[1 - \int_0^1 \frac{h(x, 1-p)\ \mathrm{d}x}{1-s-x}\right] \end{aligned} \tag{7}$$

from which it is fairly straightforward to show that

$$xh(x, p) = (1-x)h(1-x, 1-p) + W(p)x\,\delta(x) \tag{8}$$

where the weight of the delta function $W(p)$ is

$$W(p) = 1 - \int_0^1 \frac{h(x, 1-p)\ \mathrm{d}x}{1-x}. \tag{9}$$

We recognize this as the normalized conductivity (equation (3)) in the limit where $s = 1$ (i.e. $\sigma_2 = 0$) which is the dilute random resistor problem of percolation theory [19]. Thus we expect $W(1) = 1$ and $W(p) = 0$ for $p < 1 - p_c$. The delta function at the origin can be understood physically in terms of the random superconducting–normal network. At zero frequency, which corresponds to $s = 0$ in the inductor–capacitor network, the inductors have zero resistance. For $p > 1 - p_c$ the zero-resistance elements percolate and the effective conductance diverges.

The identity (8) relating $h(x, p)$ to $h(1-x, 1-p)$ is a general result that applies to *any* uncorrelated random lattice in any dimension. It also applies to infinitely interchangeable cellular materials [20, 21] but does *not* apply to most continuum problems (see appendix A) such as spherical inclusions in a host material where the geometry of the host is very different from the geometry of the inclusion.

3.2. Duality

Two-component continuum composites in 2d obey the well known reciprocity relation [22, 23]

$$\sigma_{eff}^{xx}(\sigma_1, \sigma_2)\sigma_{eff}^{yy}(\sigma_2, \sigma_1) = \sigma_1\sigma_2 \tag{10}$$

where σ_{eff}^{xx} and σ_{eff}^{yy} are the principal components of the conductivity tensor. This result (10) is independent of the geometry and is particularly useful in isotropic materials where $\sigma_{eff}^{xx} = \sigma_{eff}^{yy}$. Straley [24] has shown that for a 2d lattice problem, the reciprocity relation (10) is replaced by

$$\sigma_{eff}^{xx}(\sigma_1, \sigma_2, L)\sigma_{eff}^{yy}(\sigma_2, \sigma_1, D) = \sigma_1\sigma_2 \tag{11}$$

where $\sigma_{eff}(\sigma_1, \sigma_2, L)$ is the effective conductivity of the lattice L, and $\sigma_{eff}(\sigma_2, \sigma_1, D)$ is the effective conductivity of the *dual* lattice D, constructed by placing a bond with conductance σ_1, σ_2 perpendicular to every bond of L with conductance σ_2, σ_1. The 2d square net with random bonds (*not* sites) is self-dual and electrically isotropic, so we have the simple result

$$\sigma_{eff}(\sigma_1, \sigma_2)\sigma_{eff}(\sigma_2, \sigma_1) = \sigma_1\sigma_2 \tag{12}$$

where we have dropped the reference to the lattice in the arguments of σ_{eff}. This result can be written in the form

$$m(s, p)m(1 - s, p) = 1. \tag{13}$$

For random lattice problems, we also have the exchange relation (5), which when combined with (12) yields

$$\sigma_{eff}(\sigma_1, \sigma_2, p)\sigma_{eff}(\sigma_1, \sigma_2, 1 - p) = \sigma_1\sigma_2 \tag{14}$$

or

$$m(s, p)m(s, 1 - p) = 1 - \frac{1}{s}. \tag{15}$$

The results (14) and (15) are special to the random bond substitution problem on the 2d square net, and do *not* apply to the random site substitution problem. This is because in random site substitution there are effectively three types of bond and thus it is not possible to construct a dual lattice with σ_1 and σ_2 exchanged.

At $p = 1 - p = \frac{1}{2}$, equation (14) reduces to

$$\sigma_{eff}(\sigma_1, \sigma_2) = \sqrt{\sigma_1\sigma_2} \tag{16}$$

or

$$m(s, 1/2) = \sqrt{\frac{s - 1}{s}}\ . \tag{17}$$

From this we can derive the exact spectral function for the 2d square net with random bond substitution at percolation $p = p_c = \frac{1}{2}$,

$$h(x, 1/2) = \frac{1}{\pi}\sqrt{\frac{1 - x}{x}}. \tag{18}$$

3.3. The dilute limit

In the *low-concentration limit*, $p \ll 1$, it can be shown that for both site and bond substitution, the effective conductivity can be written as

$$\sigma_{eff}(\sigma_1, \sigma_2, p) = \sigma_1 + p\ \frac{(\sigma_2 - \sigma_1)}{p_I(\sigma_2/\sigma_1 - 1) + 1} \tag{19}$$

where p_I is determined from the initial slope of the effective conductivity when $\sigma_2 = 0$. For a 2d square-net *bond* substitution $p_I = \frac{1}{2}$, while for *site* substitution $p_I = 1 - 1/\pi = 0.682\ldots$ [25]. Rewriting (19) using the s-variable we have

$$m(s, p) = 1 - \frac{p}{s - p_I} \tag{20}$$

from which, using equation (4), we obtain the spectral function

$$h(x, p) = p\,\delta(x - p_I). \tag{21}$$

If we now consider the *high-concentration limit*, $1 - p \ll 1$, we have similarly

$$\sigma_{eff}(\sigma_2, \sigma_1, 1 - p) = \sigma_2 + (1 - p)\frac{(\sigma_1 - \sigma_2)}{p_I(\sigma_1/\sigma_2 - 1) + 1}. \tag{22}$$

We note that $m(s, p)$ is defined by normalizing σ_{eff} by σ_1, and is given by

$$m(s, p) = 1 - \frac{p_I(1 - p)}{(1 - p_I)(1 - s - p_I)} - \frac{(p - p_I)}{(1 - p_I)s} \tag{23}$$

which leads to the spectral function

$$h(x, p) = \frac{p_I(1 - p)}{(1 - p_I)}\,\delta(1 - x - p_I) + \frac{(p - p_I)}{(1 - p_I)}\,\delta(x). \tag{24}$$

The second term in equation (24) is an example of the term involving $W(p)$ in equation (8).

3.4. Effective-medium theory

An effective-medium theory for the conductivity can be constructed from the dilute results of the previous section in the usual way [26]:

$$p\frac{(\sigma_2 - \sigma_{eff})}{1 + p_I(\sigma_2/\sigma_{eff} - 1)} + (1 - p)\frac{(\sigma_1 - \sigma_{eff})}{1 + p_I(\sigma_1/\sigma_{eff} - 1)} = 0. \tag{25}$$

This leads to a quadratic equation for $m = \sigma_{eff}/\sigma_1$ which can be solved, yielding

$$m(s, p) = \frac{1}{2(1 - p_I)}\left[1 - 2p_I + \frac{(p_I - p)}{s} + \frac{\sqrt{(s - p_I)^2 + p(2s - 1)(2p_I - 1) - p(1 - p)}}{s}\right]. \tag{26}$$

When the function in the square root is negative, $m(x, p)$ will have an imaginary part, and so if we define

$$f(x, p, p_I) = p(1 - p) - (x - p_I)^2 - p(2x - 1)(2p_I - 1) \tag{27}$$

and use the definition of the spectral function (equation (4)), we extract the general form of the effective-medium spectral function:

$$h(x, p) = \begin{cases} \dfrac{\sqrt{f(x, p, p_I)}}{2\pi x(1 - p_I)} & 0 \leqslant f(x, p, p_I) \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

For random bond substitution $p_I = \frac{1}{2}$, and equation (28) has the simple form [11]

$$h(x, p) = \begin{cases} \dfrac{\sqrt{p(1-p) - (x - \frac{1}{2})^2}}{x\,\pi} & 0 \leqslant p(1-p) - (x - \frac{1}{2})^2 \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

Note that this is symmetric under the exchange $p \leftrightarrow 1 - p$ and coincides with the exact result at $p = p_I = \frac{1}{2}$, given in equation (18),

$$h(x, p_c) = \frac{1}{\pi}\sqrt{\frac{1-x}{x}}. \tag{30}$$

In effective-medium theory, the percolation threshold is $p_c = 1 - p_I$. For random bond substitution this coincides with the exact percolation threshold, $p_c = \frac{1}{2}$.

3.5. Moment expansions

It is often useful to consider the *moments* of the spectral function, which can be found from the definition (3), by doing an expansion of the denominator in the integrand to obtain

$$m(s, p) = 1 - \sum_{r=0}^{\infty} \frac{1}{s^{r+1}} \int_0^1 x^r h(x, p)\ \mathrm{d}x \tag{31}$$

or

$$m(s, p) = 1 - \sum_{r=0}^{\infty} \frac{\mu_r(p)}{s^{r+1}} \tag{32}$$

where $\mu_r(p)$ is the rth moment of the spectral function. This expansion corresponds to the *weak-scattering limit* [27], usually written as

$$\frac{\sigma_{eff}}{\sigma_1} = 1 - \sum_{r=o}^{\infty} \mu_r(p) \left(\frac{\sigma_1 - \sigma_2}{\sigma_1}\right)^r \tag{33}$$

which is useful when the contrast between the two components σ_1 and σ_2 of the composite is small, i.e. when $|(\sigma_1 - \sigma_2)/\sigma_1| = |1/s| \ll 1$. For a particular class of composites, known as infinitely interchangeable composites [20] (which includes all random lattice problems), it has been proved [28] that $\mu_r(p)$ is a polynomial in p of degree $r + 1$.

We now derive a number of recursion relations between the different moments $\mu_r(p)$. From equation (8), which is a consequence of the exchange relation equation (5), we can write

$$\int_0^1 x^{n+1} h(x, p)\ \mathrm{d}x = \int_0^1 x^n (1-x) h(1-x, 1-p)\ \mathrm{d}x \tag{34}$$

which leads to the general recursion relation

$$\mu_{n+1}(p) + (-1)^{n+1}\mu_{n+1}(1-p) = \sum_{r=1}^{n} (-1)^{r-1} \binom{n}{r-1} \mu_r(1-p). \tag{35}$$

This result (35) is true for *any* lattice with uncorrelated random substitution. The first few terms are

$$\begin{aligned}
&\mu_1(p) - \mu_1(1-p) = 0 \\
&\mu_2(p) + \mu_2(1-p) = \mu_1(1-p) \\
&\mu_3(p) - \mu_3(1-p) = \mu_1(1-p) - 2\mu_2(1-p) \\
&\mu_4(p) + \mu_4(1-p) = \mu_1(1-p) - 3\mu_2(1-p) + 3\mu_3(1-p) \\
&\mu_5(p) - \mu_5(1-p) = \mu_1(1-p) - 4\mu_2(1-p) + 6\mu_3(1-p) - 4\mu_4(1-p).
\end{aligned} \tag{36}$$

For random bond substitution where we have both duality (equation (12)) and the exchange relation (equation (5)) there is one additional set of independent recursion relations. The most useful form is derived from equation (15) and has the general form

$$\mu_n(p) + \mu_n(1-p) = \sum_{r=1}^{n} \mu_{r-1}(p)\mu_{n-r}(1-p). \tag{37}$$

The first few terms are

$$\begin{aligned}
\mu_1(p) + \mu_1(1-p) &= \mu_0(p)\mu_0(1-p) \\
\mu_2(p) + \mu_2(1-p) &= \mu_0(p)\mu_1(1-p) + \mu_1(p)\mu_0(1-p) \\
\mu_3(p) + \mu_3(1-p) &= \mu_0(p)\mu_2(1-p) + \mu_1(p)\mu_1(1-p) + \mu_2(p)\mu_0(1-p) \\
\mu_4(p) + \mu_4(1-p) &= \mu_0(p)\mu_3(1-p) + \mu_1(p)\mu_2(1-p) + \mu_2(p)\mu_1(1-p) \\
&\quad + \mu_3(p)\mu_0(1-p) \\
\mu_5(p) + \mu_5(1-p) &= \mu_0(p)\mu_4(1-p) + \mu_1(p)\mu_3(1-p) + \mu_2(1-p)\mu_2(p) \\
&\quad + \mu_3(p)\mu_1(1-p) + \mu_4(p)\mu_0(1-p).
\end{aligned} \tag{38}$$

Combining (36) with (38), it is possible to express the *odd* moments $\mu_{2n+1}(p)$ in terms of all the lower-order *even* moments, $\mu_0,\ \mu_2,\ \ldots,\ \mu_{2n}$. The first three odd moments are given by

$$\begin{aligned}
\mu_1(p) &= \frac{1}{2}\mu_0(p)\left[1-\mu_0(p)\right] \\
\mu_3(p) &= \frac{1}{8}\left[\mu_0(p)^4 - 4\mu_0(p)^3 + 5\mu_0(p)^2 - 2\mu_0(p) - 8\mu_0(p)\mu_2(p) + 12\mu_2(p)\right] \\
\mu_5(p) &= \frac{5}{2}\mu_4(p) - \mu_0(p)\mu_4(p) - \frac{1}{2}\mu_2(p)^2 - \frac{5}{2}\mu_2(p) - \frac{1}{2}\mu_0(p)^3\mu_2(p) \\
&\quad - \frac{5}{2}\mu_0(p)^2\mu_2(p) + \frac{17}{4}\mu_0(p)\mu_2(p) - \frac{1}{16}\mu_0(p)^6 + \frac{1}{2}\mu_0(p)^5 \\
&\quad - \frac{25}{16}\mu_0(p)^4 + \frac{19}{8}\mu_0(p)^3 - \frac{7}{4}\mu_0(p)^2 + \frac{1}{2}\mu_0(p).
\end{aligned} \tag{39}$$

For the *even* moments, $\mu_{2n}(p)$, it is only possible to obtain an expression for the even part of the moment, $\frac{1}{2}[\mu_{2n}(p) + \mu_{2n}(1-p)]$, in terms of the lower-order *even* moments. The first three such even moments are

$$\begin{aligned}
\frac{1}{2}\left[\mu_0(p) + \mu_0(1-p)\right] &= \frac{1}{2} \\
\frac{1}{2}\left[\mu_2(p) + \mu_2(1-p)\right] &= \frac{1}{4}\mu_0(p)\left[1-\mu_0(p)\right] \\
\frac{1}{2}\left[\mu_4(p) + \mu_4(1-p)\right] &= \frac{1}{16}[3\mu_0(p)^4 - 12\mu_0(p)^3 + 11\mu_0(p)^2 - 2\mu_0(p) \\
&\quad - 24\mu_0(p)\mu_2(p) + 12\mu_2(p)].
\end{aligned} \tag{40}$$

Obviously if all of the moments, both even and odd, could be expressed in terms of the lower moments, then the problem would be soluble in closed form, which it is not.

We can now derive the low-order moments for both site and bond substitution, using a technique that maps the expansion in p discussed in section 3.3 above to the perturbation expansion in $1/s$ [20]. In the low-concentration limit we use (21) to obtain

$$\mu_n = p p_I^n \qquad \text{for } p \ll 1. \tag{41}$$

In the high-concentration limit we use (24) to obtain

$$\mu_n = (1-p)p_I(1-p_I)^{n-1} \qquad \text{for } 1-p \ll 1. \tag{42}$$

Writing μ_n as a polynomial in p of degree $n+1$ which is correct in both limits (small p and small $(1-p)$), the zeroth moment must have the form

$$\mu_0 = p \tag{43}$$

and the higher moments for $n \geqslant 1$ are

$$\mu_n(p) = p(1-p)\frac{p_I\left[(1-p_I)^{n-1}+p_I^{n-1}\right]}{2}\left\{1+\frac{\left[(1-p_I)^{n-1}-p_I^{n-1}\right]}{\left[(1-p_I)^{n-1}+p_I^{n-1}\right]}(2p-1)\right.$$
$$\left. + \; a_4^n p(1-p) + a_5^n(2p-1)p(1-p)+\cdots\right\} \tag{44}$$

where the coefficients a_4^n and higher are as yet unknown, except that to ensure that $\mu_n(p)$ is a polynomial of degree $n+1$ we require $a_m^n = 0$ for all $m > n+1$. The polynomial (44) has been separated into terms that are even (odd) under the exchange $p \leftrightarrow 1-p$. The result (44) is a general result for all lattices where there is only one type of site and the geometry only depends on p. The quantity p_I can be determined from explicit calculations that can be done in the single-impurity limit [25]. This use of p_I makes a useful connection between the single-impurity (strong-local-scattering) and the weak-scattering limit.

3.5.1. Random bond substitution. For random bond substitution on the square net $p_I = \frac{1}{2}$, so the leading term in (44) involving $(2p-1)$ is zero. We therefore have

$$\mu_n(p) = \frac{p(1-p)}{2^n}\left[1+a_4^n p(1-p)+a_5^n(2p-1)p(1-p)+\cdots\right]. \tag{45}$$

This immediately implies that

$$\mu_0 = p \qquad \mu_1 = \frac{p(1-p)}{2} \qquad \mu_2 = \frac{p(1-p)}{4}. \tag{46}$$

The recursion relations, equation (39), give

$$\mu_3 = \frac{p(1-p)}{8}\left[1+p(1-p)\right]. \tag{47}$$

These moment results (46) and (47) for random bond substitution were previously derived by Bruno and Golden [28] using a rather different approach.

The recursion relations (39) and (40), coupled with equation (45), imply

$$\mu_4 = \frac{p(1-p)}{16}\left[1+3p(1-p)+a_5^4(2p-1)p(1-p)\right] \tag{48}$$

and

$$\mu_5 = \frac{p(1-p)}{32}\left[1+(6-a_5^4)p(1-p)+4a_5^4(2p-1)p(1-p)+(2+4a_5^4)p^2(1-p)^2\right] \tag{49}$$

where a_5^4 is an unknown constant. The moments μ_6 and μ_7 could be similarly obtained with two unknown constants.

In the discussion of the numerical simulations in section 4.3, we obtain $a_5^4 = 0.1 \pm 0.5$ by fitting equations (48) and (49) to the simulation data. This suggests that a_5^4 may in fact be equal to zero which, if true, gives the interesting result that all of the moments μ_1 to μ_5 are symmetric under the exchange $p \leftrightarrow 1-p$. This cannot be true in general for *all* of the higher moments, because if it were the recursion relations (equations (35) and (37)) would determine all of the moments, and thus we would know $m(s, p)$ exactly.

It is interesting to compare these results for $\mu_0(p)$ to $\mu_5(p)$ with the effective-medium moments by expanding equation (26) in powers of $1/s$ and setting $p_I = \frac{1}{2}$. We find that, except for μ_0, all of the moments are symmetric under the exchange $p \leftrightarrow 1-p$, and that they coincide with the exact results for $\mu_0(p)$ to $\mu_3(p)$ and with the symmetric part of $\mu_4(p)$ (which is known exactly). That is, the effective-medium values for $\mu_4(p)$ and $\mu_5(p)$ are given by equations (48) and (49) with $a_5^4 = 0$. In a diagrammatic evaluation of the moments on a three-dimensional cubic lattice, Bergman and Kantor [29] also found that μ_1 and μ_2 were given exactly by effective-medium theory.

3.5.2. Random site substitution. For random site substitution on the square net $p_I = 1 - 1/\pi$ and we have

$$\mu_n(p) = p(1-p)\frac{(\pi-1)\left[1+(\pi-1)^{n-1}\right]}{2\pi^n}\left\{1+\frac{\left[1-(\pi-1)^{n-1}\right]}{\left[1+(\pi-1)^{n-1}\right]}(2p-1)\right.$$
$$\left. + \; a_4^n p(1-p) + a_5^n(2p-1)p(1-p) + \cdots\right\}. \tag{50}$$

This immediately implies that

$$\begin{aligned}\mu_0 &= p\\ \mu_1 &= \frac{p(1-p)(\pi-1)}{\pi}\\ \mu_2 &= \frac{p(1-p)(\pi-1)}{2\pi}\left[1+\frac{(2-\pi)}{\pi}(2p-1)\right].\end{aligned} \tag{51}$$

In this case the only recursion relation is equation (35) which unfortunately gives no further information about μ_3. We do know that μ_3 has the form

$$\mu_3 = \frac{p(1-p)(\pi^2-2\pi+2)(\pi-1)}{2\pi^3}\left[1+\frac{(2-\pi)\pi}{(\pi^2-2\pi+2)}(2p-1)+a_4^3 p(1-p)\right] \tag{52}$$

and fitting equation (52) to the numerical data gives $a_4^3 = 0.14 \pm 0.07$, as discussed in section 4.3.

We can compare these results for $\mu_0(p)$ to $\mu_3(p)$ with the effective-medium moments which can be obtained by expanding equation (26) in powers of $1/s$ and setting $p_I = 1 - 1/\pi$. We find that the effective-medium values for $\mu_0(p)$ to $\mu_2(p)$ coincide with the exact results (equation (51)) and that the effective-medium value for $\mu_3(p)$ is given by equation (52) with $a_4^3 = (10\pi - 2\pi^2 - 10)/(\pi^2 - 2\pi + 2) = 0.3001\ldots$. For the *site* problem the effective-medium theory is not symmetric under the exchange $p \leftrightarrow 1-p$.

3.5.3. General remarks. In a lattice problem where there are M inequivalent impurity bonds (sites) the conductivity in the dilute limit may be written as

$$m(s,p) = 1 - \frac{1}{M}\sum_i \frac{p}{s-p_I^i} \tag{53}$$

where the sum is over the M different types of bond (site) and $-1/(1-p_I^i)$ would be the initial slope of the conductivity curve if vacant bonds (sites) of the ith type only were present. When properly averaged over all types of bond (site), the initial slope of the conductivity curve is

$$\frac{-1}{M}\sum_i \frac{1}{1-p_I^i}.$$

In the *weak-scattering limit*, $|s| \gg 1$, we can expand equation (53) to obtain

$$m(s, p) = 1 - \frac{p}{s} - \frac{p}{s^2}\langle p_I\rangle - \frac{p}{s^3}\left\langle p_I^2\right\rangle + \cdots. \tag{54}$$

where the angular bracket denotes an average over the different types of impurity bonds, i.e.

$$\left\langle p_I^n\right\rangle = \frac{1}{M}\sum_i (p_I^i)^n.$$

There is a similar result in the high-concentration limit, $1 - p \ll 1$, where

$$m(s, p) = 1 - \frac{p}{s} - \frac{1-p}{s^2}\langle p_I\rangle - \frac{1-p}{s^3}\langle p_I(1-p_I)\rangle + \cdots\ . \tag{55}$$

From these two equations, (54) and (55), we can develop a form for the moment polynomials similar to equation (44):

$$\mu_n(p) = p(1-p)\frac{\left\langle p_I(1-p_I)^{n-1}\right\rangle + \langle (p_I)^n\rangle}{2}\left[1 + \frac{\left\langle p_I(1-p_I)^{n-1}\right\rangle - \langle p_I{}^n\rangle}{\left\langle p_I(1-p_I)^{n-1}\right\rangle + \langle p_I{}^n\rangle}(2p-1)\right.$$
$$\left. + a_3^n p(1-p) + \cdots\right]. \tag{56}$$

In particular, the leading terms are

$$\begin{aligned}
\mu_0 &= p \\
\mu_1 &= p(1-p)\langle p_I\rangle \\
\mu_2 &= \frac{1}{2}p(1-p)\langle p_I\rangle\left[1 + \frac{\langle p_I\rangle - 2\left\langle p_I^2\right\rangle}{\langle p_I\rangle}(2p-1)\right].
\end{aligned} \tag{57}$$

We note again that even though the p_I^i are determined from the single-impurity strong-scattering limit ($s = 1$), they determine the low-order terms of the weak-scattering expansion, up to μ_2, without any calculation of defect interactions. For continuum systems, determining μ_2 requires the calculation of the interaction between defect pairs, as discussed in appendix A.

3.6. Critical properties

The property of the random resistor network that has received the most attention from physicists is the behaviour of the effective conductivity at the percolation threshold [30]. If the host has $\sigma_1 = 1$ and the defect is $\sigma_2 = 0$, then there is a percolation threshold p_c such that the effective conductivity is zero for $p \geqslant p_c$ and non-zero for $p < p_c$. Near the percolation threshold the effective conductivity scales as

$$\sigma_{eff} \sim (p_c - p)^t \qquad \text{for } (p_c - p) \ll 1 \tag{58}$$

where t is the conductivity exponent. Noting that $\sigma_2 = 0$ corresponds to $s = 1$, we can see from equation (3) that this critical behaviour must be controlled by the spectral function near $x = 1$ where the denominator of the integrand is very small. There has been some previous work [14, 16] on the scaling behaviour of the spectral function, but it was hindered by a lack of detailed knowledge of the spectral function.

It is convenient to start the discussion of the scaling behaviour within the effective-medium approximation and then extend the ideas to the more general result. In effective-medium theory there is a percolation threshold at $p_c = 1 - p_I$. For $p < p_c$ the spectral

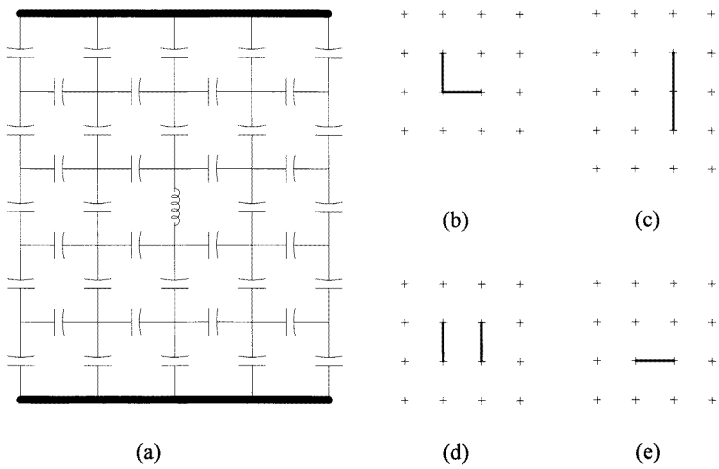

Figure 1. The geometry of the network with *bond* substitution. (a) A single defect in a network with $L = 4$. The dark lines represent perfectly conducting busbars. (b), (c), (d) The most strongly interacting pairs of defects where the solid line indicates the location of the defect bonds. For the orientation shown, the single defect (e) does not couple to the applied potential and makes no contribution to the spectral function.

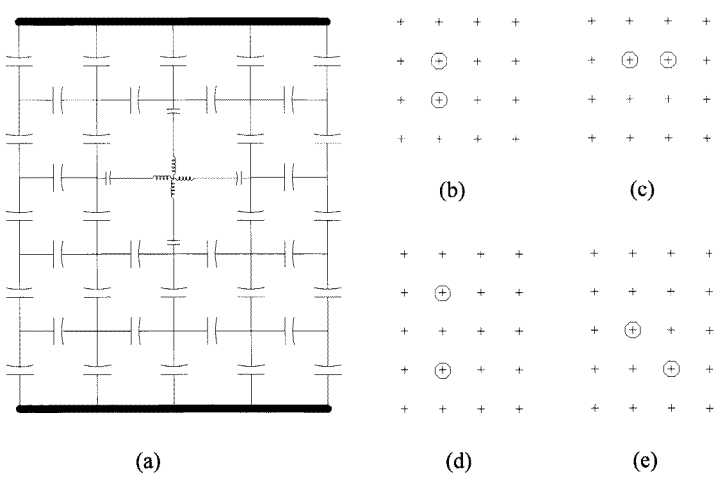

Figure 2. The geometry of the network with *site* substitution. The small capacitors have conductance σ_1, the small inductors have capacitance σ_2, and the large capacitors have conductance $\sigma_1/2$. (a) A single defect in a network with $L = 4$. The dark lines represent perfectly conducting busbars. (b)–(e) The most strongly interacting pairs of defects where the hexagons indicate the locations of the defect sites.

function does not fill the entire $[0, 1]$ interval and there is a gap Δ in the spectrum such that $h(x, p) = 0$ for $x > 1 - \Delta$. As p approaches p_c this gap decreases and goes to zero as

$$\Delta \sim (p_c - p)^2. \tag{59}$$

For $x < 1 - \Delta$ the spectral function has the form

$$h(x, p) \sim \sqrt{(1 - x - \Delta)} \qquad \text{for } (1 - x - \Delta) \ll 1. \tag{60}$$

We introduce a new variable $y = 1 - x - \Delta$, and break the integral representation of the normalized conductivity, equation (3), into two parts:

$$m(1, p) \sim 1 - \int_0^{\Delta} \frac{\sqrt{y}}{y + \Delta}\, \mathrm{d}y - \int_{\Delta}^{U} \frac{h(y, p)}{y + \Delta}\, \mathrm{d}y. \tag{61}$$

The second integral is insensitive to the small Δ and is of order one. The first integral depends strongly on Δ and we obtain

$$m(1, p) \sim \sqrt{\Delta}. \tag{62}$$

Thus, within effective-medium theory we have $m(1, p) \sim (p_c - p)$. This result, that the effective-medium conductivity exponent is $t = 1$, is well known [19] and can be obtained directly from equation (26).

More generally, we would expect the gap (59) to scale with an exponent different from two and that near p_c there will be some exponent η such that the spectral function has the form

$$h(x, p) \sim (1 - x - \Delta)^{\eta} \tag{63}$$

near the edge of the gap. To give the correct scaling behaviour for σ_{eff}, we must have $\Delta \sim (p_c - p)^{t/\eta}$. In 2d we have the exact form of the spectral function at p_c for random bond substitution on the square net (equation 18), and by universality we expect the same behaviour for all lattices. This would indicate that $\eta = \frac{1}{2}$ in 2d and that $\Delta \sim (p_c - p)^{2t}$ where $t = 1.3$ is the conductivity exponent [31] in 2d. In the numerical simulations presented later we found evidence of the gap Δ, but the method did not enable us to determine the critical exponents.

4. Simulations

We have calculated the effective conductance of the random networks with a very efficient algorithm [18] that uses the Y–Δ transformation to calculate the equivalent conductance of circuits in 2d. The simulations were performed on a square lattice of L by $(L + 1)$ bonds with perfectly conducting busbars placed along the two shorter edges. The conductances of the remaining bonds are assigned randomly, according to the random bond *or* the random site substitution rule, and the equivalent conductance between the two busbars is calculated. An example of the network with $L = 4$ and one defect is shown for bond substitution in figure 1(a) and for site substitution in figure 2(a), where we have drawn networks of inductors and capacitors to emphasize the concept of the spectral function as a density of states. The results presented are for $L = 64$, averaged over 32 realizations at each probability p. All of our simulation results were checked against simulations on lattices of up to $L = 256$, and no significant differences were found. We set $\sigma_1 = 1$ and determine σ_2 from s where x, the real part of s, is chosen at 240 equally spaced values across the interval $(-0.2, 1.2)$ and ϵ, the imaginary part of s, has some small positive value, the exact value of which is not very important. The spectral function is then determined from equation (4). The finite value of ϵ broadens the poles into Lorentzians and thus smooths the final spectral function. The smaller the value of ϵ, the greater the resolution of the structure of the spectral function, and the greater the number of values of x that are needed to give an accurate representation. A value of $\epsilon = 0.003$ was found to give acceptable results, partly determined by the agreement with the sum rule $\mu_0 = p$ given in equations (46) and (51).

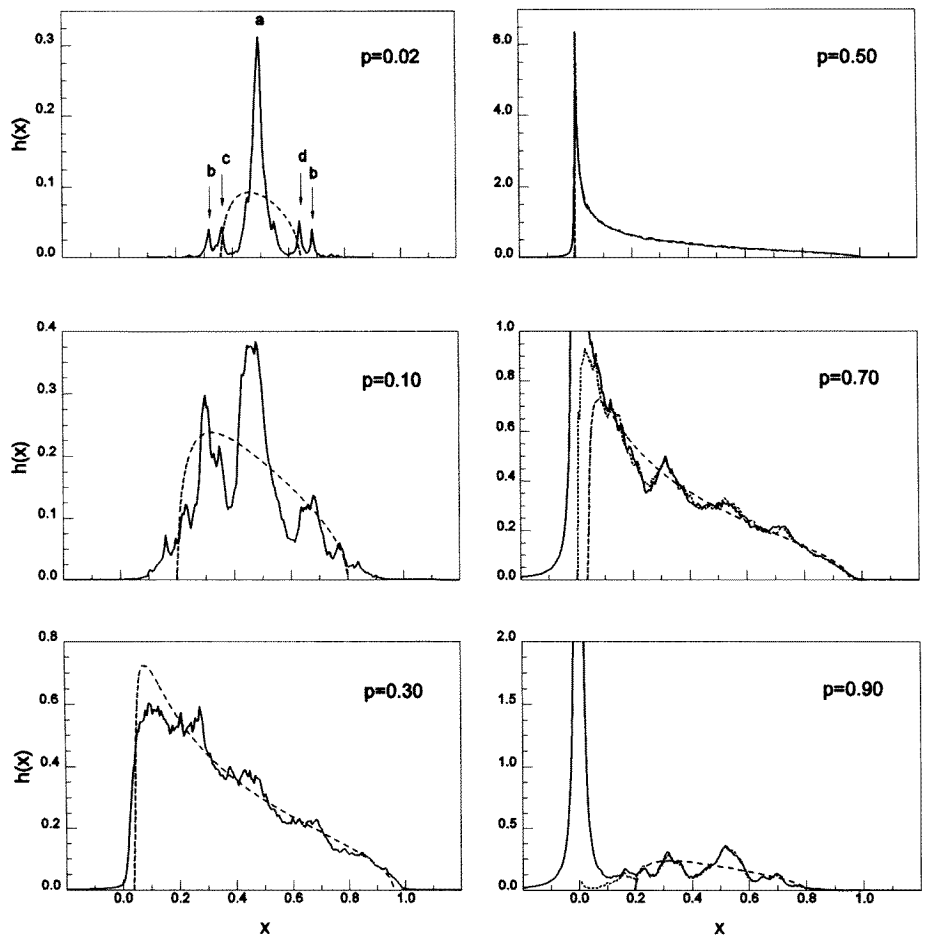

Figure 3. The spectral function $h(x)$ for a two-dimensional square net with random *bond* disorder at different defect concentrations p. The solid lines show data from the simulations. The dashed lines are for the effective-medium theory, equation (29). For $p > 0.5$ the dotted lines show the spectral function at $(1-p)$ transformed via equation (8). Note that the horizontal scale is the same for all graphs but the vertical scales are different.

4.1. Random bond substitution

The spectral functions for random bond substitution at $p = 0.02$, $p = 0.1$, $p = 0.3$, $p = p_c = 0.5$, $p = 0.7$ and $p = 0.9$ are shown in figure 3. In all of the panels the solid line shows the calculated spectral function and the dashed line is for the effective-medium theory, equation (29). For $p \ll 1$ the defects are essentially isolated and the spectral function is a single peak centred at $x = \frac{1}{2}$. Interpreting this as a network of capacitors with conductance $\sigma_1 = \mathrm{i}\omega C$ and a few isolated inductors with impedance $\sigma_2 = 1/(\mathrm{i}\omega L)$, this corresponds to a resonance at $\omega = 1/\sqrt{LC}$ with the corresponding

$$x = 1/[1 + (\omega^2 LC)^{-1}] = \frac{1}{2}.$$

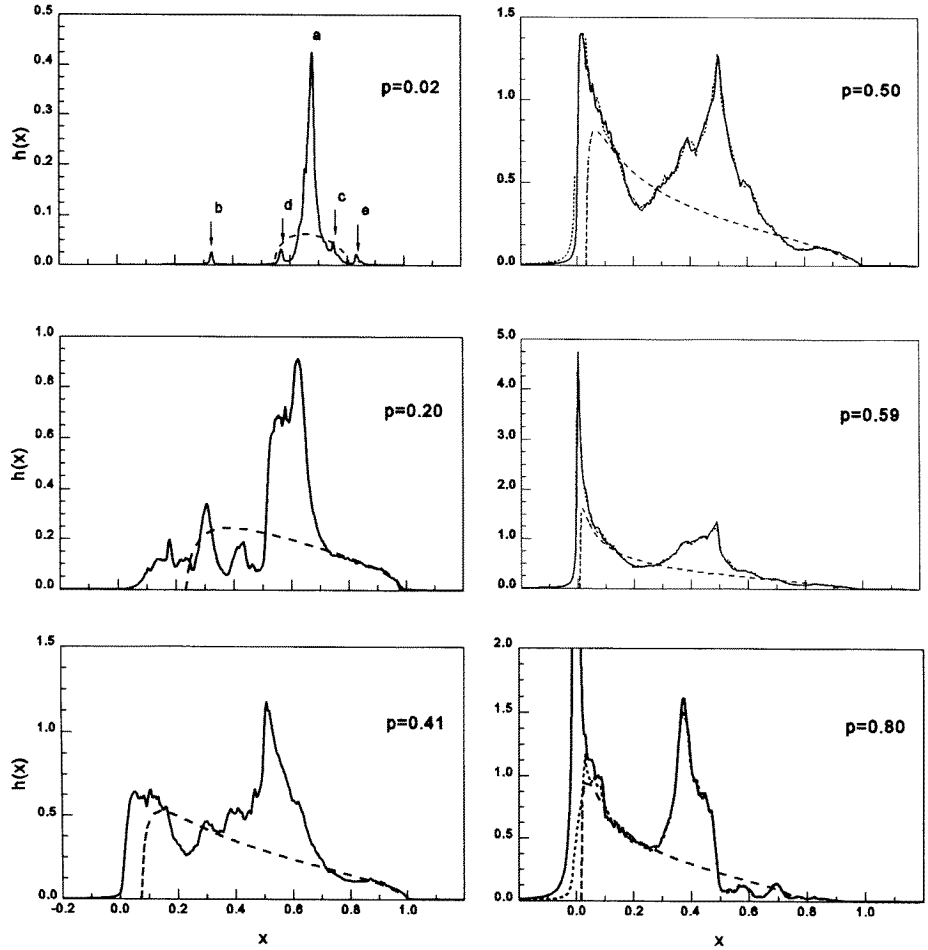

Figure 4. The spectral function $h(x)$ for a two-dimensional square net with random *site* disorder at different defect concentrations p. The host percolates for $p \leqslant p_c = 0.41$; the defects percolate at $p \geqslant 1 - p_c = 0.59$. The solid lines show data from the simulations. The dashed lines are for the effective-medium theory, equation (28), with $p_I = 1 - 1/\pi$. For $p > 0.5$ the dotted lines show the spectral function at $(1 - p)$ transformed via equation (8). Note that the horizontal scale is the same for all graphs but the vertical scales are different.

The width of the peak in the spectral function is determined entirely by the value of ϵ. As p is increased other peaks, resulting from interacting defects, appear. In the first frame of figure 3, we indicate the position of the resonances associated with the four most strongly interacting pairs of defects and the associated peaks in the spectral function can be clearly seen. The labels b to d correspond to the labels of the defects illustrated in figure 1. Note that the positioning of these resonances is symmetric about $x = \frac{1}{2}$ because the square lattice is self-dual and c defects are the dual of d defects. The b defects are self-dual and two resonances are symmetrically placed about $x = \frac{1}{2}$. Note that the orientation of the defects is important; defects a, c and d make no contribution to the density of states when the defects are oriented parallel to the busbars. In this sense the calculated spectral function is like a

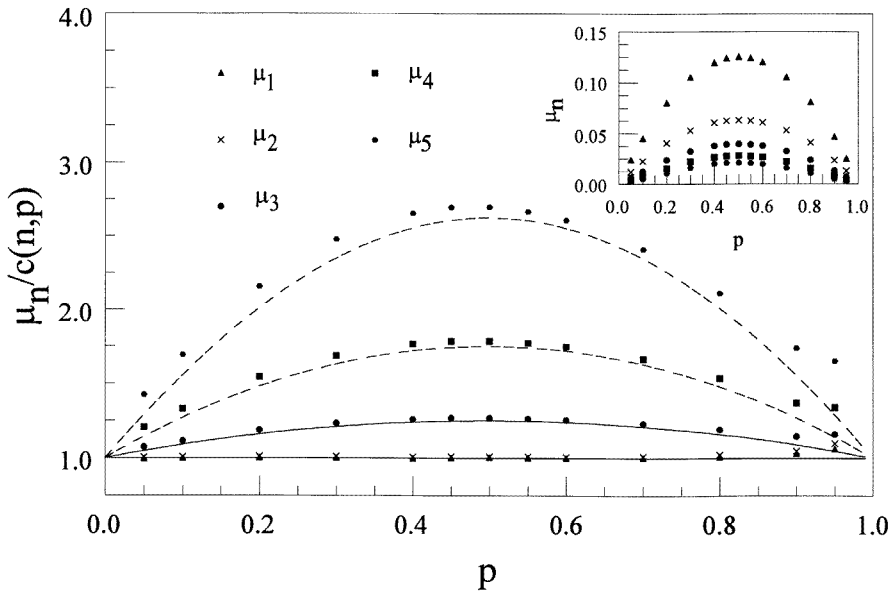

Figure 5. The moments $\mu_n(p)$ of the spectral function, scaled by the prefactor $c(n, p) = (\frac{1}{2})^n p(1-p)$ for a two-dimensional square net with random *bond* disorder. The inset shows the unscaled moments to show the overall shape of the moment curve. The solid lines show the exact theory for $\mu_1(p)$ to $\mu_3(p)$ and the dashed lines show the theory for μ_4 and μ_5 with one unknown parameter, which we have chosen to be zero.

projected density of states, only detecting those states that couple to the given boundary conditions (an applied external field). However, averaging over many samples is equivalent to averaging each sample over both orientations, so the final spectral function is that of the isotropic material and is like the full density of states. Note that the width of the central peak at $p = 0.02$ in figure 3 is broadened by both the finite ϵ *and* by more distant weakly interacting pairs.

As p is increased the spectral function expands to fill more of the interval $[0, 1]$, and loses much of its structure, approaching the smooth curve of the effective-medium theory. At $p = p_c = \frac{1}{2}$ effective-medium theory coincides with the exact result, equation (18). The simulations agree very well with the theory and show the square-root edge at $x = 1$, suitably smoothed because of the finite ϵ, and the square-root divergence at $x = 0$. Note that although we have the exact form for the spectral function at $p = p_c = \frac{1}{2}$ this provides no information about the critical behaviour: the critical behaviour is controlled by the way the edge of the spectral function approaches $x = 1$, which is clearly very difficult to extract from the simulations.

For $p \geqslant \frac{1}{2}$ we plot $h(x, p)$ evaluated directly, and evaluated from the transformation $[(1-x)/x]h(1-x, 1-p)$ as given via equation (8). The transformed curve is a good fit to the direct curve and the contribution of the delta function (suitably broadened) can also be clearly seen. The vertical scale is chosen to show the structure away from the origin. This scale truncates the curve at the origin which peaks at a value of about 40 for $p = 0.90$. The weight of the delta function $W(p)$, determined from the simulations, is found to decrease approximately linearly from $W(1) = 1$, to $W(\frac{1}{2}) = 0$, as expected from equation (9). We note that in all the figures the non-zero value of $h(x, p)$ for $x < 0$ and $x > 1$ is just a result

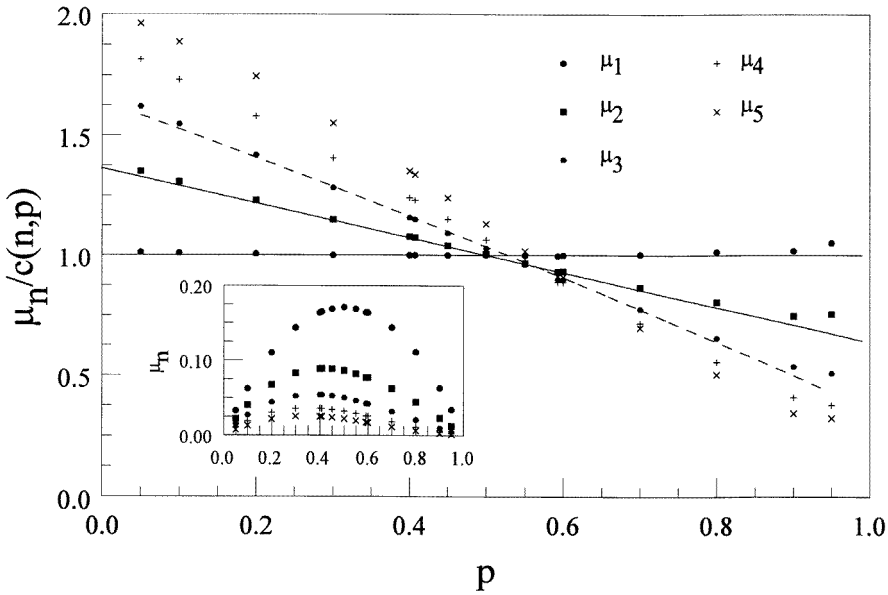

Figure 6. The moments $\mu_n(p)$ of the spectral function, scaled by the prefactor $c(n, p) = ((\pi - 1)^n + 1)p(1 - p)/(2\pi^n)$ for a two-dimensional square net with random *site* disorder. The inset shows the unscaled moments to show the overall shape of the moment curve. The solid lines show the exact theory for $\mu_1(p)$ and $\mu_2(p)$. The dashed line shows the theory for μ_3 with one fitted parameter.

of the small finite value of ϵ.

4.2. Random site substitution

The spectral functions for random site substitution at $p = 0.02$, $p = 0.2$, $p = p_c = 0.407$, $p = 0.5$, $p = 1 - p_c = 0.593$, and $p = 0.8$ are shown in figure 4. Note that p is the fraction of *defects* but we refer to the percolation threshold as the point where the *host* just spans the system and thus $p_c = 0.407$ and $1 - p_c = 0.593$. In all of the panels the solid line shows the calculated spectral function and the dashed line shows the effective-medium theory spectral function, equation (28), with $p_I = 1 - 1/\pi$. The discussion below is similar to that of the random bond substitution problem but there are some important differences, most notably because there is no duality relation (12) for the random site problem.

For $p \ll 1$ the defects are essentially isolated and the spectral function is a single peak centred at $x = p_I = 1 - 1/\pi$. We interpret this as a network of capacitors with impedance $\sigma_1 = \mathrm{i}\omega C$ with a few isolated site defects of the type shown in figure 2(a). This corresponds to a resonance at $\omega = \sqrt{(\pi - 1)/LC}$. As p is increased, other peaks, resulting from interacting defects, appear. In the first frame of figure 4 we indicate the position of the resonances associated with the four most strongly interacting pairs of defects, and the associated peaks of the spectral function can be seen. The labels b to e correspond to the labels of the defects illustrated in figure 2. There may be more than one resonance associated with each type of defect but only those most widely separated from the single-defect peak are marked. Note that the spectral function is not symmetric about $x = \frac{1}{2}$, even in the dilute limit, when there is no symmetry based on duality. Once again, the orientation

of the defects relative to the busbars is relevant: defects b and c have the same structure but different orientation and contribute to different resonances.

As p approaches p_c the spectral function expands to fill the interval $[0, 1]$, and the total weight increases. However, unlike in the random bond problem, most of the structure is retained and the agreement with effective-medium theory does *not* improve as p_c is approached. This indicates that the smoothness of the spectral function and the agreement between the simulations and effective-medium theory in the bond case is a very special result, following from the self-dual nature of the square lattice in the bond problem. Note that at p_c, the spectral function still appears to have a square-root edge singularity at $x = 1$, which is expected from universality. The spectral function does not diverge at $x = 0$.

For $p \geqslant 0.5$ we plot $h(x, p)$ evaluated directly and evaluated from the transformation $[(1-x)/s]h(1-x, 1-p)$ as given in equation (8). The transformed curve is a good fit to the direct curve, and at $p = 0.8$ the contribution of the broadened delta function at $x = 0$ can be clearly seen. Note that, although there is a sharp peak at $x = 0$ when $p = 1 - p_c = 0.59\ldots$, the weight of the delta function at the origin is zero. A plot of the weight of the delta function $W(p)$, determined from the simulations, is a curve that decreases from $W(1) = 1$, with an initial slope of $-1/(1-p_I)$, to $W(1-p_c) = 0$, as expected from equation (9). Again we note that the non-zero value of $h(x, p)$ for $x < 0$ and $x > 1$ is just a result of the small finite value of ϵ.

4.3. Moments

In figure 5 we plot the moments of the spectral function for random *bond* substitution, evaluated by numerical integration of the curves presented in figure 3. The inset is a plot of μ_1 to μ_5 and it is clear that the shape of the moment curves is dominated by the prefactor of equation (45),

$$c(n, p) = p(1-p)/2^n$$

and that the moments decrease rapidly with increasing n. In the main body of the figure we have plotted the moments scaled by the prefactor $c(n, p)$. The lines are the theoretical curves from section 3.5: the solid lines for μ_1 to μ_3 are known exactly and given by equations (46) and (47). We see that the simulations are an excellent fit to the theory. The systematic deviation near $p = 1$ occurs because of the contribution from the broadened delta function at $x = 0$. The dashed lines for μ_4 and μ_5 are for equations (48) and (49) with the parameter a_5^4 chosen to be zero (see the discussion in section 3.5).

In figure 6 we plot the moments of the spectral function for random *site* substitution, evaluated by numerical integration of the curves presented in figure 4. The inset is a plot of μ_1 to μ_5 and although the shape of the moment curves is dominated by the prefactor of equation (50),

$$c(n, p) = p(1-p)[(\pi-1)^n + 1]/(2\pi^n)$$

the contribution of the term proportional to $(2p-1)$ is clear. Again the moments decrease rapidly with increasing n. In the main body of the figure we have plotted the moments scaled by the prefactor $c(n, p)$. The lines are the theoretical curves from section 3.5: the solid lines for μ_1 to μ_2 are known exactly and given by equation (51). The dashed line for μ_3 is for equation (52) with one fitted parameter, $a_4^3 = 0.144$. We see that the simulations are an excellent fit to the theory with a systematic deviation near $p = 1$ resulting from the contribution of the broadened delta function at $x = 0$. The data indicate that the term in the braces of equation (50) that is linear in $(2p-1)$ is the most important, and that the higher-order coefficients a_m^n must be very small.

5. Conclusions

We have completed a study of the spectral function for 2d random resistor networks with square-net geometry. These systems were chosen as they are ideally suited for large-scale computer simulations, because the Y–Δ transformation can be used. This leads to much more detailed results for the spectral function, at all compositions, than have ever been obtained previously. We have shown that the spectral function can be regarded as a density of states, and we have given the first five moments of the bond and site versions of the two-component resistor network.

Acknowledgments

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Appendix A. Some related continuum results

It is instructive to review some previous results for continuum systems in the spirit of the lattice results discussed in this paper. For a two-component composite in d dimensions the weak-scattering results give [27]

$$\mu_0 = p \qquad \mu_1 = \frac{p(1-p)}{d} \tag{A1}$$

for all volume fractions p of inclusions and are independent of the *shape* of the inclusions. In the low-concentration limit, the spectral function has the form

$$h(x, p) = p\frac{1}{M_\alpha}\sum_i \alpha_i\, \delta(x - p_I^i) \tag{A2}$$

where the sum is over the M_α types of inclusion (see section 3.5) and enough orientations to ensure that the composite is isotropic. For example, elliptical inclusions must be averaged over two perpendicular orientations. In general, the α_i and the p_I^i are very difficult to calculate but the results are known for a few special cases. Equation (A2) implies the following sum rules:

$$\frac{1}{M_\alpha}\sum_i \alpha_i = 1 \qquad \frac{1}{M_\alpha}\sum_i \alpha_i\, p_I^i = \frac{1}{d} \tag{A3}$$

and that the higher-order moments will have the form

$$\mu_n(p) = \frac{p}{M_\alpha}\sum_i \alpha_i (p_I^i)^n + \mathrm{O}(p^2). \tag{A4}$$

For continuum problems, the geometry is not determined solely by the volume fraction p and the low-concentration limit is *not* equivalent to the high-concentration limit with σ_1 and σ_2 exchanged. For example, a uniform matrix with conductivity σ_1 containing a volume fraction of $p \ll 1$ of isolated spheres with conductivity σ_2 is *not* equivalent to a matrix with a high concentration of overlapping spheres with conductivity σ_1 and a small volume fraction of interstitial regions with conductivity σ_2. Nevertheless, it is interesting to consider

the high-concentration limit and extract some information about the moments. The spectral function in this limit has the form

$$h(x, p) = \left[p - \frac{(1-p)}{M_\beta} \sum_i \beta_i \right] \delta(x) + \frac{(1-p)}{M_\beta} \sum_i \beta_i \, \delta(x - q_I^i) \tag{A5}$$

where the coefficient of the delta function at the origin is chosen to get the zeroth moment μ_0 correct, and the β_i and q_I^i have no simple relation to the α_i and p_I^i. The delta function at the origin makes no contribution to the higher-order moments so we have

$$\mu_1(p) = \frac{1-p}{M_\beta} \sum_i \beta_i q_I^i = \frac{1-p}{d} \tag{A6}$$

and for $n \geqslant 2$

$$\mu_n(p) = \frac{1-p}{M_\beta} \sum_i \beta_i (q_I^i)^n + \mathrm{O}(1-p)^2. \tag{A7}$$

To get both the low-concentration limit, $p \ll 1$, and the high-concentration limit, $1-p \ll 1$, correct, the general form for the moments must be

$$\mu_n(p) = p(1-p) a_2^n \left\{ 1 + a_3^n (2p-1) + a_4^n p(1-p) + \cdots \right\} \tag{A8}$$

where $a_m^n = 0$ for $m > n+1$. The most important feature of equation (A8) is that for all $n \geqslant 1$, the moments μ_n are proportional to $p(1-p)$.

In general $\mu_1(p)$ is the highest known moment [27], and is independent of the shape of the inclusion. Unfortunately, unlike in the lattice problems discussed in this paper, knowing the single-defect solution does not provide enough information to calculate $\mu_2(p)$, and it is necessary to calculate the scattering from interacting defects, i.e. to calculate the low-concentration conductivity correct to order p^2. This has been done for circular inclusions in 2d [32] and spherical inclusions in 3d [33, 21, 34]. Then, making the assumption that $\mu_n(p)$ is a polynomial in p of degree $n+1$, we can use the methods of section 3.5 to obtain $\mu_2(p)$ valid for all p. Circular or spherical inclusions are not examples of infinitely interchangeable materials so there is no rigorous proof [28] that $\mu_n(p)$ is a polynomial in p of degree $n+1$, but it is probably correct, and we assume it to be so. In 3d this leaves μ_2 as the highest moment known [20] but in 2d we can use the reciprocity relationship [22, 23] to obtain $\mu_3(p)$.

Following Djordjević *et al* [32] we have that for circular inclusions in the low-concentration limit

$$\mu_2(p) = \frac{p}{4} - \frac{5}{12} p^2 + \mathrm{O}(p^3). \tag{A9}$$

Writing this in the form of equation (A8) we obtain the result, valid for all p:

$$\mu_2(p) = \frac{p(1-p)}{6} \left[1 - \frac{1}{2}(2p-1) \right]. \tag{A10}$$

The reciprocity relationship is

$$m(s, p) m(1-s, p) = 1 \tag{A11}$$

which is formally identical to equation (13). This implies a recursion relationship from which we can generate the exact third moment:

$$\mu_3 = \frac{p(1-p)}{12} \left[1 - \frac{1}{2}(2p-1) - \frac{1}{2} p(1-p) \right]. \tag{A12}$$

Note that there is no relationship equivalent to equation (15) for this problem.

To obtain any higher moments using this method would require calculating the effect of more than two interacting defects and would be extremely difficult. From these moments μ_0, μ_1, μ_2, and μ_3, we can generate the weak-scattering result for σ_{eff} using equation (33).

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Note that the notation for *site* problems used in this paper is slightly different from that used here. For site problems the defect bond has conductance $\sigma_{defect} = \sigma_1\sigma_2/(\sigma_1 + \sigma_2)$.
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